Amalgamated Worksheet # 1 Solutions

Various Artists

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For all exercises, V is a finite dimensional complex vector space

1.) Prove that if $T \in \mathcal{L}(V)$ has only one eigenvalue, then every vector $v \in V$ is a generalized eigenvector of T (*Hint*: Use the Jordan decomposition of T).

Solution Let n be the dimension of V and let λ be the only eigenvalue of T. Therefore, the Jordan decomposition of T results in the expression

$$V = \text{Null}(T - \lambda I)^n.$$

Therefore, if $v \in V$, $v \in \text{Null}(T - \lambda I)^n$ so v is a generalized eigenvector of T.

2.) For this problem, suppose that S and T are operators on a finite dimensional complex vector space V.

a.) Suppose that ST is nilpotent. Prove that TS is nilpotent.

Solution: Suppose $(ST)^k = 0$. We note that

$$(TS)^{k+1} = TSTS \cdots TS(k + 1 \text{ times}) = T(ST)^k S = T \cdot 0 \cdot S = 0$$

so TS is nilpotent.

b.) Suppose S and T are nilpotent and $ST = TS$. Prove that $S + T$ is nilpotent.

Solution: We recall the binomial theorem, which states that for complex numbers z and w and any positive integer j,

$$(z + w)^k = \sum_{j=0}^{k} \binom{k}{j} z^j w^{k-j}.$$

If you examine the proof of the binomial theorem, you will see that every step is true if x and y are replaced with linear operators which commute (commutativity is very important here!). To this end, assume that $S^m = 0$ and $T^n = 0$ for positive integers n and m, we note that:

$$(S+T)^{n+m} = \sum_{j=0}^{m+n} \binom{k}{j} S^j T^{m+n-j}.$$

If $j \geq m$ then $S^j = 0$, and if $j \leq m$ then $m+n-j \geq n$ so $T^{m+n-j} = 0$. In any case, every term of the sum vanishes so ST is nilpotent.

c.) Suppose S and T are nilpotent. Must $S+T$ be nilpotent? Give a proof or give a counterexample.

Solution This is false. To see this, let V be two-dimensional and $\beta = (v_1, v_2)$ a basis for V. Let S and T be the linear operators such that

$$\mathcal{M}_\beta(S) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \mathcal{M}_\beta(T) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easily seen that both matrices square to zero so S and T are nilpotent. Furthermore,

$$\mathcal{M}_\beta(S+T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is seen to square to the identity. Therefore, $S+T$ is not nilpotent.

3.) Let V be an $n-$dimensional complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, ..., \lambda_m$ be the (distinct) eigenvalues of T (hence $m \leq n$). We know from class that if $U_k = \text{Null}(T - \lambda_k I)^n$, we have

$$V = U_1 \oplus \cdots \oplus U_m.$$

a.) Prove that each U_k is invariant under T.

Solution: Using that T commutes with both itself and $\lambda_k I$, we see that T commutes $T - \lambda_k I$ and hence with $(T - \lambda_k I)^n$. Let $v \in U_k$. This means $(T - \lambda_k I)^n v = 0$. Therefore

$$(T - \lambda_k I)^n T v = T(T - \lambda_k I)^n v = T(0) = 0$$

which implies $Tv \in U_k$. Therefore U_k is T invariant.

b.) Prove that $T - \lambda_k I$ restricted to U_k is nilpotent.

Solution: We first note that U_k is invariant under $T - \lambda_k I$ as it invariant under both T and $\lambda_k I$. If $v \in U_k$, then by definition $v \in \text{Null}(T - \lambda_k I)^n$, so $(T - \lambda_k I)^n(v) = 0$.

This implies that $(T-\lambda_k I)^n$ is the zero operator on U_k so $(T-\lambda_k I)|_{U_k}$ is nilpotent.

c.) Consider $E_i \in \mathcal{L}(V)$ defined by $E_i(v_1+v_2+\cdots+v_m) = v_i$ whenever $v_k \in U_k$ (notice that this is well defined by the direct sum decomposition). Prove that T commutes with each E_i.

Solution: Let v in V and write v uniquely as

$$v = v_1 + \cdots v_m \text{ for } v_i \in U_i.$$

Then we have $TE_i(v) = T(v_i)$ and

$$E_i T(v) = E_i(T(v_1) + \cdots + T(v_m)) = T(v_i)$$

where we have used $T(v_k) \in U_k$ (part *a.*)). This shows $TE_i = E_i T$.

d.) Use the $E_i's$ to show that we can write $T = D + N$ where D is diagonalizable and N is nilpotent with $DN = ND$.

solution: We first start with two easy lemmas:

Lemma 1: $\sum_{i=1}^m E_i = I$

Proof: Let $v \in V$ and as in part *c.*), write $v = v_1 + \cdots + v_m$ with $v_i \in U_i$. Then we have

$$\left(\sum_{i=1}^m E_i\right)(v) = \sum_{i=1}^m E_i(v) = \sum_{i=1}^m v_i = v$$

Lemma 2: $E_i E_j = 0$ if $i \neq j$.

Proof: Writing $v = v_1 + \cdots + v_m$ as above, we have

$$E_i E_j(v) = E_i(v_j) = 0$$

Now we move on to the problem. We define

$$D = \sum_{i=1}^m \lambda_i E_i \text{ and } N = \sum_{i=1}^m (T - \lambda_i I)E_i = \sum_{i=1}^m (TE_i - \lambda_i E_i)$$

We first see that

$$N + D = \sum_{i=1}^m (TE_i - \lambda_i E_i + \lambda_i E_i) = T \cdot \left(\sum_{i=1}^m E_i\right) = T \cdot I = T.$$

It is easily verified that $E_i^2 = E_i$. Using this, lemma 2, and part c.) above, we see from the expressions of D and N that D and N must commute with each other.

If $v \in U_k$ then by definition,

$$D(v) = \sum_{i=1}^{m} \lambda_i E_i(v) = \lambda_k E_k(v) = \lambda_k v$$

so v is an eigenvector of D. Finding bases of each U_k and concatenating them to form a basis for V produces a basis of V consisting of eigenvectors of D. Therefore D is diagonalizable.

Finally, the fact that $E_i^2 = E_i$, along with lemma 2 and part c.) above, implies that

$$N^n = \sum_{i=1}^{m} (T - \lambda_i I)^n E_i.$$

Writing $v = v_1 + \cdots v_m$ as above, we see that

$$(T - \lambda_i I)^n E_i(v) = (T - \lambda_i I)^n (v_i) = 0$$

since $v_i \in U_i = \mathrm{Null}(T - \lambda_i I)^n$. Therefore each $(T - \lambda_i I)^n E_i$ is zero, implying $N^n = 0$ so N is nilpotent.

2 Peyam Tabrizian

Problem 1:

Find all the generalized eigenvectors of $T \in \mathcal{L}(\mathbb{R}^3)$ defined by:

$$T(x, y, z) = (x + y + z, y + z, z)$$

Solution: The matrix A with respect to the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $\mathbb{R}^3$ is:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Since A is upper-triangular, by Prop. 5.18, A has only one eigenvalue, $\lambda = 1$.

Now use the following method:

Method: To find all the generalized eigenvectors of T, for each eigenvalue λ you found, find a basis for $Nul((T - \lambda I)^n)$, where $n = dim(V)$.

Here, all we need to find is a basis for $Nul((A - I)^3)$.

But:

$$(A-I)^3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence:

$$Nul\left((A-I)^3\right) = Nul \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Which means that:

$$Nul((T-I)^3) = Span\left\{(1,0,0),(0,1,0),(0,0,1)\right\} = \mathbb{R}^3$$

Hence **any** vector in $\mathbb{R}^3$ is a generalized eigenvector of T.

Problem 2:

Suppose that $T \in \mathcal{L}(V)$ has n distinct eigenvalues (where $n = dim(V)$), and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (but not necessarily with the same eigenvalues). Show that $ST = TS$.

Solution: Let $\lambda_1, \cdots, \lambda_n$ be the distinct eigenvalues of T and let $v_1, \cdots, v_n$ be the corresponding eigenvectors of T, so $\boxed{T(v_i) = \lambda_i v_i}$ (for $i = 1, \cdots n$). However, since $v_1, \cdots, v_n$ are also eigenvectors of S (by assumption), we know that $\boxed{S(v_i) = \mu_i v_i}$ for (possibly different) eigenvalues μ_i ($i = 1, \cdots, n$).

Then by Theorem 5.6, we know that the list $(v_1, \cdots, v_n)$ is linearly independent. Hence $(v_1, \cdots, v_n)$ is a linearly independent list of n vectors in the $n-$dimensional vector space V. Hence $(v_1, \cdots, v_n)$ spans V.

Now let v be an arbitrary vector in V. We want to show $\boxed{ST(v) = TS(v)}$.

However, since $v \in V$ and $(v_1, \cdots, v_n)$ spans V, we know that there exist scalars $a_1, \cdots, a_n$ such that $v = a_1v_1 + \cdots + a_nv_n$.

Then:

$$\begin{aligned} ST(v) =& ST(a_1v_1 + \cdots + a_nv_n) \\ =& S(a_1T(v_1) + \cdots + a_nT(v_n)) \\ =& S(a_1\lambda_1v_1 + \cdots + a_n\lambda_nv_n) \\ =& a_1\lambda_1S(v_1) + \cdots + a_n\lambda_nS(v_n) \\ =& a_1\lambda_1\mu_1v_1 + \cdots + a_n\lambda_n\mu_nv_n \end{aligned}$$

But also:

$$\begin{aligned} TS(v) =& TS(a_1v_1 + \cdots + a_nv_n) \\ =& T(a_1S(v_1) + \cdots + a_nS(v_n)) \\ =& T(a_1\mu_1v_1 + \cdots + a_n\mu_nv_n) \\ =& a_1\mu_1T(v_1) + \cdots + a_n\mu_nT(v_n) \\ =& a_1\mu_1\lambda_1v_1 + \cdots + a_n\mu_n\lambda_nv_n \\ =& a_1\lambda_1\mu_1v_1 + \cdots + a_n\lambda_n\mu_nv_n \\ =& ST(v) \end{aligned}$$

Note: It would also have been enough to show that $ST(v_i) = TS(v_i)$ for all $i = 1, \cdots, n$ and then invoked the uniqueness-part of the **linear extension lemma**.

Problem 3:

Show that if V is a vector space over $\mathbb{C}$ and if 0 is the only eigenvalue of $T \in \mathcal{L}(V)$, then T is nilpotent

Solution: By theorem 8.23, V can be expressed as a direct sum of the generalized eigenspaces of T $(*)$. However, since T has only one eigenvalue (namely 0), T has only one generalized eigenspace (the one corresponding to the eigenvalue $\lambda = 0$), and hence by $(*)$, we have that V equals to the generalized eigenspace of T corresponding to $\lambda = 0$. However, by Corollary 8.7, that eigenspace is precisely $Nul(T^n)$, where $n = dim(V)$. Therefore, we have $\boxed{V = Nul(T^n)}$.

Now if $v \in V$, then $v \in Nul(T^n)$, so $T^n(v) = 0$, and since v was arbitrary, we get $\boxed{T^n = 0}$. That is, T is nilpotent. $\square$

Problem 4:

Show that if $Nul(T-\lambda I) = Nul((T-\lambda I)^2)$ for all λ, then V has a basis of eigenvectors of T (that is, T, is *diagonalizable*)

From Corollary 8.25, we know that V has a basis $(v_1, \cdots, v_n)$ consisting of generalized eigenvectors of T. Let $\lambda_1, \cdots, \lambda_n$ be the corresponding eigenvalues.

<u>Goal:</u> Show that each v_i is actually an eigenvector of T ($i = 1, \cdots, n$)

Fix $i = 1, \cdots, n$.

Our assumption with $\lambda = \lambda_i$ implies that $Nul(T-\lambda_i I) = Nul((T-\lambda_i I)^2)$, and hence by Prop 8.5 (with $T - \lambda_i I$ instead of T) implies that:

$$Nul(T - \lambda_i I) = Nul((T - \lambda_i I)^2) = \cdots = Nul((T - \lambda_i I)^n)$$

Now since v_i is a generalized eigenvector corresponding to λ_i, v_i is in $Nul((T-\lambda_i I)^n)$ by Corollary 8.7. Hence by what we've just shown, $v_i \in Nul((T - \lambda_i I))$, that is v_i is an **eigenvector** of T corresponding to λ_i.

Hence $(v_1, \cdots, v_n)$ is actually a basis of V consisting of eigenvectors of T (and not just generalized eigenvectors) $\square$

Problem 5:

(if time permits) Suppose $T \in \mathcal{L}(V)$

(a) Show that $T(T - \lambda I)^n = (T - \lambda I)^n T$.

(b) Use (a) to show that $(T - \lambda I)(T - \mu I)^n = (T - \mu I)^n (T - \lambda I)$.

Solution:

(a) By the binomial formula, we have:

$$
\begin{aligned}
T(T-\lambda I)^n &= a_i T\left(\sum_{i=0}^{n} T^i(-\lambda I)^{n-i}\right) \\
&= \sum_{i=0}^{n} a_i T T^i (-\lambda I)^{n-i} \\
&= \sum_{i=0}^{n} a_i T^{i+1}(-\lambda)^{n-i} I \\
&= \sum_{i=0}^{n} a_i (-\lambda)^{n-i} T^{i+1} I \\
&= \sum_{i=0}^{n} a_i (-\lambda)^{n-i} T^{i+1} \\
&= \sum_{i=0}^{n} a_i (-\lambda)^{n-i} I^{n-i} T^{i+1} \\
&= \sum_{i=0}^{n} a_i (-\lambda I)^{n-i} T^i T \\
&= \left(\sum_{i=0}^{n} a_i (-\lambda I)^{n-i} T^i\right) T \\
&= \left(a_0(-\lambda I)^n + a_1(-\lambda I)^{n-1} T + \cdots + a_{n-1}(-\lambda I) T^{n-1} + a_n T^n\right) T \\
&= \left(a_n T^n + a_{n-1}(-\lambda I) T^{n-1} + \cdots + a_1(-\lambda I)^{n-1} T + a_0(-\lambda I)^n\right) T \\
&= \left(a_0 T^n + a_1(-\lambda I) T^{n-1} + \cdots + a_{n-1}(-\lambda I)^{n-1} T + a_n + (-\lambda I)^n\right) T
\end{aligned}
$$

(This follows because actually $a_0 = a_n$, $a_1 = a_{n-1}$, etc., by looking at the formula f

$$
\begin{aligned}
&= \left(\sum_{i=0}^{n} a_i (-\lambda I)^{i} T^{n-i}\right) T \\
&= (-\lambda I + T)^n T \\
&= (T-\lambda I)^n T
\end{aligned}
$$

(b)

$$
\begin{aligned}
(T-\lambda I)(T-\mu I)^n &= T(T-\mu I)^n - \lambda I (T-\mu I)^n \\
&\overset{(a)}{=} (T-\mu I)^n T - \lambda (T-\mu I)^n \\
&= (T-\mu I)^n T - (T-\mu I)^n (\lambda I) \\
&= (T-\mu I)^n (T-\lambda I)
\end{aligned}
$$